Hybrid resonance: Maxwell with a changing sign permittivity tensor

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The issues

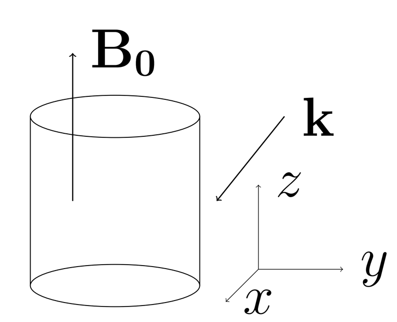
- tensor $\underline{\underline{\varepsilon}}$ with diagonal coefficient $\alpha \simeq rx$ changing sign in the domain $\Omega = (-1, 1)$
- no unicity of the solution
- singular fields

Our method

- regularize adding viscosity $\nu > 0$
- construct quasi-solutions
- get a well-posed limit variational formulation

Resonant heating

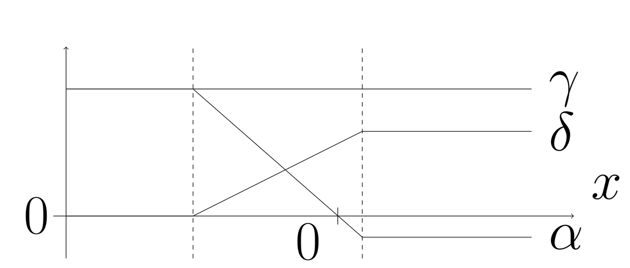
Objective: study the propagation of a wave in a tokamak that enters in resonance with the plasma.



Wave-particle model: Maxwell for EM, Newton for electron dynamics. For $\mathbf{B} = \nabla \wedge \mathbf{E}$, once linearized, the harmonic problem in time writes

$$\nabla \wedge \mathbf{B} - \begin{pmatrix} \alpha & i\delta & 0 \\ -i\delta & \alpha & 0 \\ 0 & 0 & \gamma \end{pmatrix} \mathbf{E} = 0 \quad \text{in } \Omega$$

$\delta(x) > 0$, $\gamma = cst > 0$
Fourier transform in z: $\nabla \to (\partial_x, 0, ik_z)^t$.



Reduces to ODE on $\mathbf{u} = (E_y, B_y)^t$ with Robin BC

$$\begin{cases} -\mathsf{I}_\gamma \mathbf{u}'' + \frac{\mathsf{N}}{\alpha}\mathbf{u} = 0, & \mathsf{I}_\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1/\gamma \end{pmatrix} \\ \mathsf{I}_\gamma \mathbf{u}'(\pm 1) \mp \begin{pmatrix} i\sigma & 0 \\ 0 & i/\sigma \end{pmatrix} \mathbf{u}(\pm 1) = \mathbf{f}(\pm 1) \end{cases}$$

for $\sigma > 0$, $\mathbf{f}$ $\mathbb{C}^2$-valued. Adding **viscosity** $\alpha \to \alpha + i\nu$ regularizes, allows to compute the heating $\nabla \cdot \mathbf{\Pi}^\nu = \nu \|\mathbf{E}^\nu\|_2^2 \underset{\nu \to 0}{\not\to} 0$. Difficulty: discretizing $\forall \mathbf{v} \in H^1(\Omega)^2$

$$b^\nu(\mathbf{u}, \mathbf{v}) = \int_\Omega \left(\mathbf{u}' \cdot \mathsf{I}_\gamma \overline{\mathbf{v}}' + \mathbf{u} \cdot \frac{\mathsf{N}^\nu \overline{\mathbf{v}}}{\alpha + i\nu} \right) dx + \text{BC} = \ell(\mathbf{v}) \qquad (\text{VF}_\nu)$$

leads to a competition between ν and δx. And for $\nu \to 0$ this problem becomes **ill-posed**.

An energy relation obtained from a family of manufactured solutions

Construct non-homogeneous quasi-solutions $\mathbf{F}^\nu \simeq \mathbf{E}^\nu$ and $\mathbf{C}^\nu \simeq \mathbf{B}^\nu$ of $\nabla \wedge \mathbf{C}^\nu - \underline{\underline{\varepsilon}}^\nu \mathbf{F}^\nu = \mathbf{g}^\nu$ and $\mathbf{C}^\nu - \nabla \wedge \mathbf{F}^\nu = \mathbf{q}^\nu$. Poynting vector of the scaled difference field gives the integral relation

$$\int_\Omega \text{Im}\left((\mathbf{E}^\nu - s\mathbf{F}^\nu) \times \overline{(\mathbf{B}^\nu - s\mathbf{C}^\nu)} \right) \cdot \nabla \varphi dx + \int_\Omega \text{Im}\left((\mathbf{E}^\nu - s\mathbf{F}^\nu) \cdot \mathbf{g}^\nu - \overline{(\mathbf{B}^\nu - s\mathbf{C}^\nu)} \cdot \mathbf{q}^\nu \right) \varphi dx = \nu \int_\Omega |\mathbf{E}^\nu - s\mathbf{F}^\nu|^2 \varphi dx \geq 0, \quad \text{for } \varphi \in \mathcal{C}^1_{0,+}.$$

The **singularity** we face is of type $1/\alpha$ as

$$E_x^\nu = -\frac{i\delta}{\alpha + i\nu} E_y^\nu - \frac{ik_z}{\alpha + i\nu} B_y^\nu,$$

so the trick is to define $F_x^\nu = 1/(\alpha + i\nu)$ and to compensate the quasi-singular terms $1/(\alpha + i\nu)$ by the derivative of $\frac{1}{r}\left(\frac{\log(r^2x^2+\nu^2)}{2} - i\,\text{atan}(\frac{rx}{\nu}) \right)$. And for the integral relation to pass to the limit each term must be L^1 independently of ν.

Note $\mathbf{w}_1^\nu = (F_y^\nu, C_y^\nu)^t \simeq \mathbf{u}^\nu$ and $\mathbf{w}_2^\nu = (C_z^\nu, F_z^\nu)^t$. The quadratic form on (E_y, B_y, s) at the limit writes

$$\mathcal{J}^+(\mathbf{u}, s) = -\text{Im} \int_\Omega (\mathbf{u} - s\mathbf{w}_1^+) \cdot \overline{(\mathsf{I}_\gamma \mathbf{u}' - s\mathbf{w}_2^+)} \varphi' dx + \text{Im} \int_\Omega (s\mathbf{z}_1^+ \cdot \overline{(\mathbf{u} - s\mathbf{w}_1^+)} - s\mathbf{z}_2^+ \cdot \overline{(\mathbf{u}' - s\mathsf{I}_{\frac{1}{\gamma}} \mathbf{w}_2^+)}) \varphi dx.$$

Formulation at $\nu = 0^+$

Define a **mixed variational formulation** associated to the minimization of this quadratic form at the limit on $(\mathbf{u}, s) \in V = H^1(\Omega)^2 \times \mathbb{C}$ under the constraint of weak Maxwell's equations on $Q = H^1(\Omega)^2 \cap \{\mathbf{v}, \mathsf{N}(0)\mathbf{v}(0) = 0\}$:

$$\begin{cases} a^+((\mathbf{u}, s), (\mathbf{v}, t)) - \overline{b((\mathbf{v}, t), \boldsymbol{\lambda})} = 0, & \forall (\mathbf{v}, t) \in V \\ b((\mathbf{u}, s), \boldsymbol{\mu}) = \ell(\boldsymbol{\mu}), & \forall \boldsymbol{\mu} \in Q \end{cases} \qquad (\text{MVF}_+)$$

with $\text{Im}\, a^+((\mathbf{u}, s), (\mathbf{v}, t)) = 0 \Leftrightarrow d\mathcal{J}^+_{(\mathbf{u},s)}(\mathbf{v}, t) = 0$. Then use classical theory of mixed variational formulations and Fredholm operator theory to get:

Theorem

There exists a unique solution $(\mathbf{u}^+, s^+) \in V$ and $\boldsymbol{\lambda}^+ \in Q$ of (MVF_+). Moreover $\mathbf{u}^+$ is the H^1 weak limit of $\mathbf{u}^\nu$, solution of (VF_ν), as $\nu \to 0^+$.

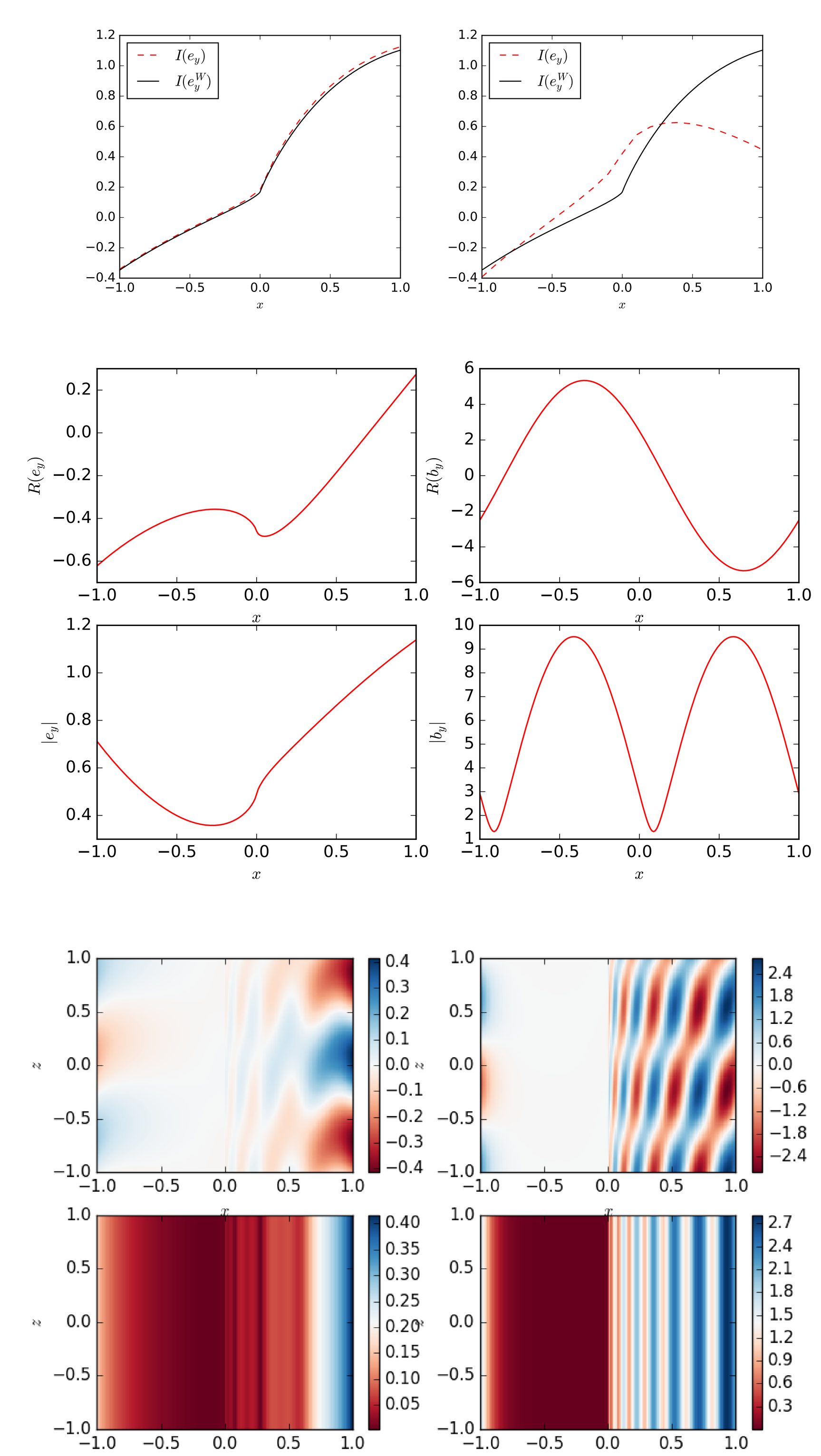


Figure 1: Whittaker test case.

Figure 1 First line: comparisons of approximations of $\text{Im}\, E_y^+$ via (MVF_+) and of $\text{Im}\, E_y^\nu$ via (VF_ν) $\nu = 10^{-7}$ to the analytical solution in black, $k_z = 0$. 1D and 2D plots: real parts above and modulus below of approximations of $E_y^+ e^{ik_z z}$ and $B_y^+ e^{ik_z z}$ via (MVF_+), with $k_z = 0$ in 1D, $k_z = 4$ in 2D.

Figure 2 Real parts above and modulus below of approximations of fields $E_x^+ e^{ik_z z}$, $E_y^+ e^{ik_z z}$ and $B_y^+ e^{ik_z z}$ with $k_z = \frac{6\pi}{\sqrt{2}}$ for the disctretization of (MVF_+).

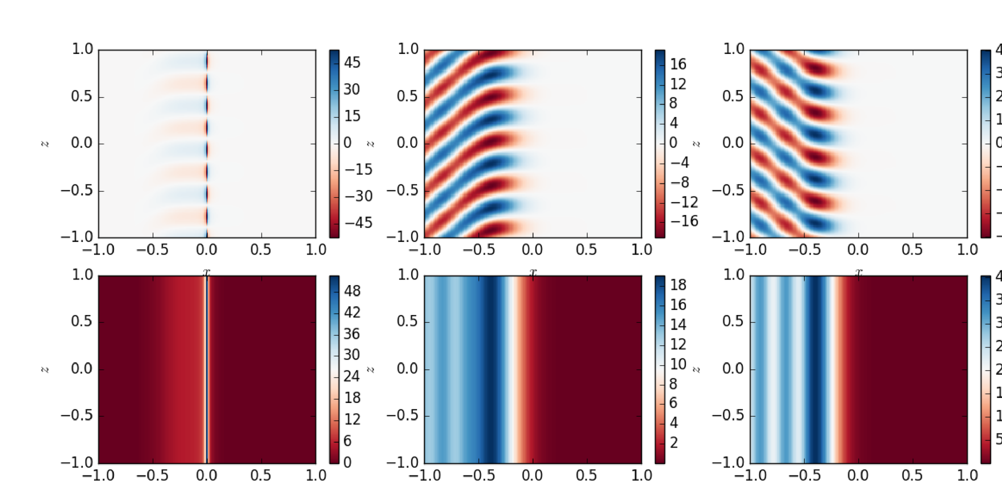


Figure 2: Case of propagative plasma for $\hat{\mathbf{u}} e^{ik_z(x+z)}$ on the left, resonant in 0, with homogeneous BC on the right.

Formulation for $\nu \to 0^+$

Similar formulation $a^+ \to a^\nu$, $Q \to Q^\nu$. As $\frac{\mathsf{N}^\nu}{\alpha + i\nu}$ is now invertible, for a continuous formulation, $Q^\nu = \{\mathbf{v} \in H^1(\Omega)^2, \Gamma^\nu(\mathbf{v}) = 0\}$ for $\Gamma^\nu(\mathbf{v}) = \nu \int_\Omega \left(\frac{1}{\alpha^2 + \nu^2} \mathbf{v} \cdot \begin{pmatrix} \delta^2 & \delta k_z \\ \delta k_z & k_z^2 \end{pmatrix} \overline{\mathbf{w}_1^\nu} + \mathbf{v} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \overline{\mathbf{w}_1^\nu} \right) dx.$

Non-local criteria: work has to be done to get a Lagrange P1 description.

References

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