Corners and cross-points in DDM formulations of Helmholtz equation

Bruno Després (LJLL-SU), Anouk Nicolopoulos (UZH) and Bertrand Thierry (CNRS, LJLL-SU)

Corners: an algebraic approach

Aborbing boundary condition Look for relations at corners Q_{kl} that complement (BC), verified at order 2 by plane wave of direction d_η: comes down to combinations of scalar products.

Then insert these corner conditions into the weak formulation of (BC), which leads to the definition of a global operator $R : u \in L^2(\Gamma) \mapsto \varphi \in L^2(\Gamma)$ such that $\varphi \in H^1_{\mathrm{br}}(\Gamma)(\simeq (\imath\omega)^{-1}\partial_{\mathbf{n}} u)$ is the solution to

$$\sum_{k=1}^{K}\int_{\Gamma_k}\left(\varphi_k\overline{\psi_k} + \frac{1}{2\omega^2}\partial_{\mathbf{t}_k}\varphi_k\overline{\partial_{\mathbf{t}_k}\psi_k}\right)\mathrm{d}\gamma + \sum_{\substack{k=1\\ l=k+1}}^{K}\left(\alpha_{kl}(\varphi_k+\varphi_l)\overline{(\psi_k+\psi_l)} + \beta_{kl}(\varphi_k-\varphi_l)\overline{(\psi_k-\psi_l)}\right)(Q_{kl}) = \sum_{k=1}^{K}\int_{\Gamma_k} u\overline{\psi_k}\mathrm{d}\gamma \qquad \forall\psi \in H^1_{\mathrm{br}}(\Gamma), \tag{1}$$

with $\alpha_{kl}, \beta_{kl} \in \imath\mathbb{R}$.

Series of properties: R is well-defined, $\|R\|_{\mathcal{L}(L^2(\Gamma))} \le 1$, $R \ne R^*$, $R+R^* > 0$, for Ω a K-sided regular polygonal domain approximating the disc of radius r as $K \to \infty$, the ABC converges towards

$$\left(1 - \frac{1}{2\omega^2 r^2}\partial_\theta^2 - \frac{\imath}{2\omega r}(1+\partial_\theta^2)\right)\partial_r u - \imath\omega u = 0.$$

Transmission conditions Adapting this ABC into a TC leads to

$$\begin{cases} -\Delta u_i^{p+1} - \omega^2 u_i^{p+1} = f, & \Omega_i,\\ (\partial_{\mathbf{n}_i} - \imath\omega T_i)\, u_i^{p+1} = -\left(\partial_{\mathbf{n}_j} + \imath\omega T_j^*\right) u_j^p, & \partial\Omega_i\cap\partial\Omega_j,\\ (\partial_{\mathbf{n}_i} - \imath\omega T_i)\, u_i^{p+1} = 0, & \partial\Omega_i\cap\Gamma. \end{cases} \tag{2}$$

The algorithm for $f = 0$ is endowed to a decreasing energy

$$E^p = \sum_i\sum_j \int_{\partial\Omega_i\cap\partial\Omega_j} (T_i + T_i^*)^{-1}\left(\partial_{\mathbf{n}_i}u_i^p - \imath\omega T_i u_i^p\right)\overline{\left(\partial_{\mathbf{n}_i}u_i^p - \imath\omega T_i u_i^p\right)} \to_p 0.$$

Key property for the proof: the isometry

$$|||\partial_{\mathbf{n}_i}u_i + \imath\omega T_i^* u_i||| = |||\partial_{\mathbf{n}_i}u_i - \imath\omega T_i u_i||| \tag{3}$$

in a given norm associated to the spectral decomposition of $T_i + T_i^*$.

Setting

Initial problem: 2D Helmholtz with Sommerfeld radiation condition

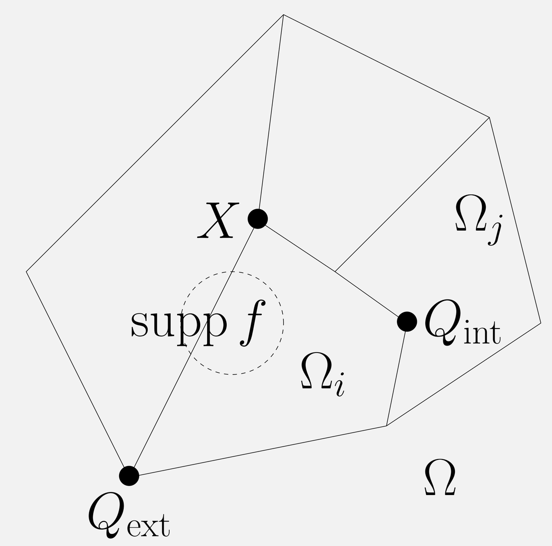
$$-\Delta u(\mathbf{x}) - \omega^2 u(\mathbf{x}) = f(\mathbf{x}),$$
$$\lim_{\|\mathbf{x}\|\to\infty}\|\mathbf{x}\|^{1/2}\left(\nabla u(\mathbf{x})\cdot\frac{\mathbf{x}}{\|\mathbf{x}\|} - \imath\omega u(\mathbf{x})\right) = 0.$$

Objective: derive a DDM with 2nd order absorbing boundary condition

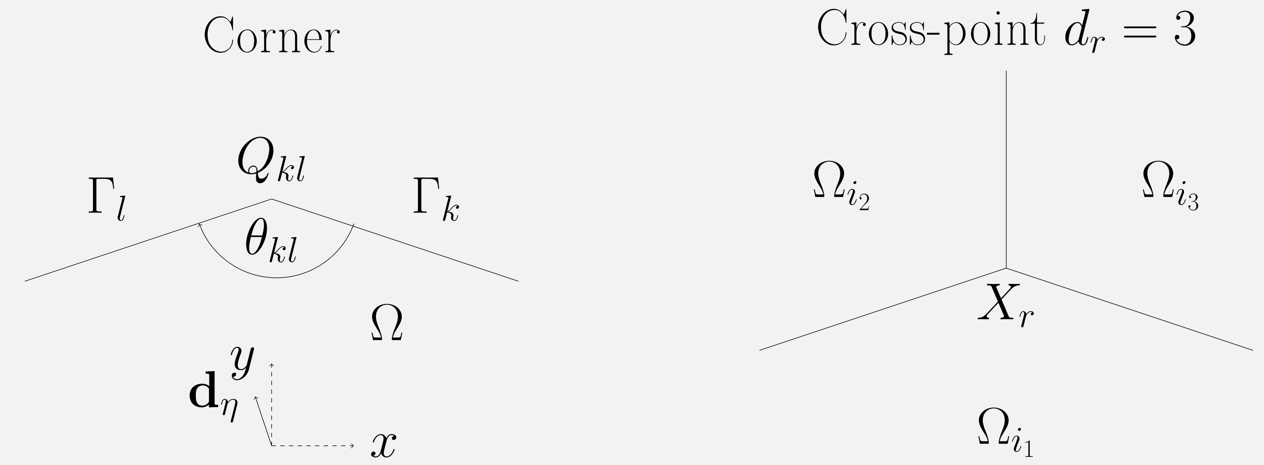
$$(1 - \frac{1}{2\omega^2}\partial_{\mathbf{t}}^2)\partial_{\mathbf{n}} u - \imath\omega u = 0 \quad \text{(BC)}$$

2nd order Robin type transmission conditions where $T \simeq (1 - \frac{1}{2\omega^2}\partial_{\mathbf{t}}^2)^{-1}$

$$(\partial_{\mathbf{n}} - \imath\omega T)u_\Sigma^{p+1} = -(\partial_{\mathbf{n}} + \imath\omega T)\Pi u_\Sigma^p \quad \text{(TC)}$$



Problem: corners like Q, cross-points like X.



Notations: compactly supported source $\operatorname{supp} f \subset \Omega$; non-overlaping domain decomposition $\cup\Omega_i$; exterior boundary $\Gamma := \partial\Omega$; oriented interfaces $\Sigma_{ij} := \partial\Omega_i \cap \partial\Omega_j$; skeleton $\Sigma := \cup_{ij}\Sigma_{ij}$; natural exchange operator Π, s.t. $(\Pi v)|_{\Sigma_{ij}} = v|_{\Sigma_{ji}}$.

Two originalities:

- transmission operators $T \ne T^*$ by contrast to work by P. Joly et al., X. Claeys and E. Parolin
- each DDM is endowed to a decreasing global energy on the skeleton

Cross-points: a more abstract frame

Similarily to what was developed for the corners, we work with cross-points conditions of type

$$\partial_{\boldsymbol{\tau}}\varphi_r + A^r\varphi_r = 0, \text{ with } A^r \in \imath\mathcal{M}_{2d_r}(\mathbb{R}),$$

where φ_r is a vector gathering the unknowns standing for $(\imath\omega)^{-1}\partial_{\mathbf{n}} u$ on each side of the d_r interfaces intersecting in X_r. We will also require that $A^r = \imath H^r$ for a symmetric matrix H^r, and define an operator $T : u \in L^2(\Sigma) \mapsto \varphi \in L^2(\Sigma)$ such that $\varphi \in H^1_{\mathrm{br}}(\Sigma)$ is the solution to

$$\sum_{i,j=1}^{N}\int_{\Sigma_{ij}}\left(\varphi_{ij}\overline{\psi_{ij}} + \frac{1}{2\omega^2}\partial_{\mathbf{t}_i}\varphi_{ij}\overline{\partial_{\mathbf{t}_i}\psi_{ij}}\right)\mathrm{d}\gamma + \frac{1}{2\omega^2}\sum_{r=1}^{N_X}(A^r\varphi_r, \psi_r)_{\mathbb{C}^{2d_r}} = \sum_{i,j=1}^{N}\int_{\Sigma_{ij}} u_i\overline{\psi_{ij}}\mathrm{d}\gamma \qquad \forall\psi \in H^1_{\mathrm{br}}(\Sigma). \tag{4}$$

Again, one has that T is well-defined, $\|T\|_{\mathcal{L}(L^2(\Sigma))} \le 1$, $T \ne T^*$, $T+T^* > 0$, and one can define an $H^1_{\mathrm{br}}(\Sigma)$ equivalent norm based on the spectral decomposition of $T+T^*$, and get a similar isometry to (3).

Under the compatibility assumption that $\Pi T \Pi = T^*$, the DDM writes

$$\begin{cases} -\Delta u_i^{p+1} - \omega^2 u_i^{p+1} = f, & \Omega_i,\\ (\partial_{\mathbf{n}} - \imath\omega T)\, u_\Sigma^{p+1} = -\Pi\left(\partial_{\mathbf{n}} + \imath\omega T^*\right) u_\Sigma^p, & \Sigma,\\ (\partial_{\mathbf{n}} - \imath\omega)\, u_\Gamma^{p+1} = 0, & \Gamma, \end{cases} \tag{5}$$

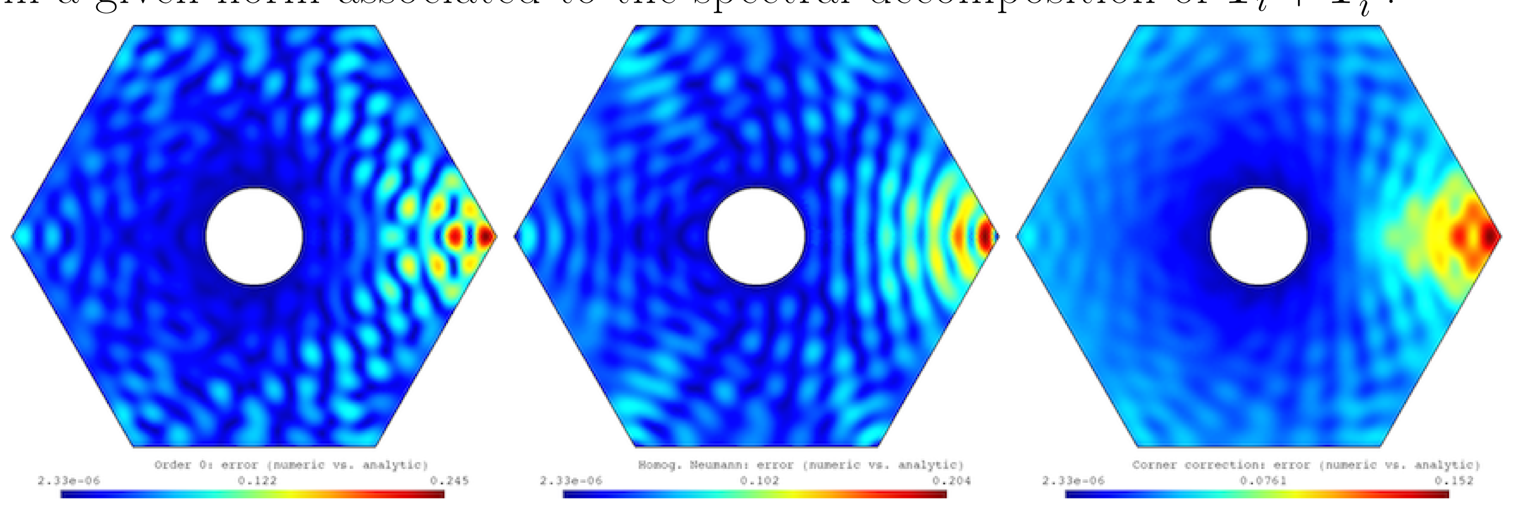
and it is endowed to the descreasing energy for $f = 0$

$$E^p := |||(\partial_{\mathbf{n}} - \imath\omega T)u_\Sigma^p|||^2 \to_p 0.$$

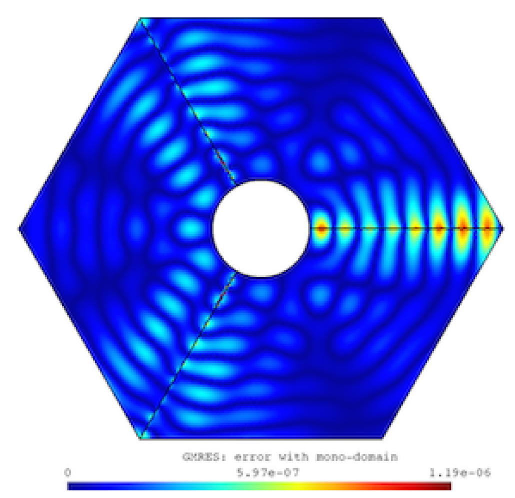
Using this proves convergence of the under-relaxed DDM, *i.e.* for $\alpha \in (0,1)$

$$(\partial_{\mathbf{n}} - \imath\omega T)\, u_\Sigma^{p+1} = -\alpha\Pi\left(\partial_{\mathbf{n}} + \imath\omega T^*\right) u_\Sigma^p + (1-\alpha)\left(\partial_{\mathbf{n}} - \imath\omega T\right) u_\Sigma^p.$$

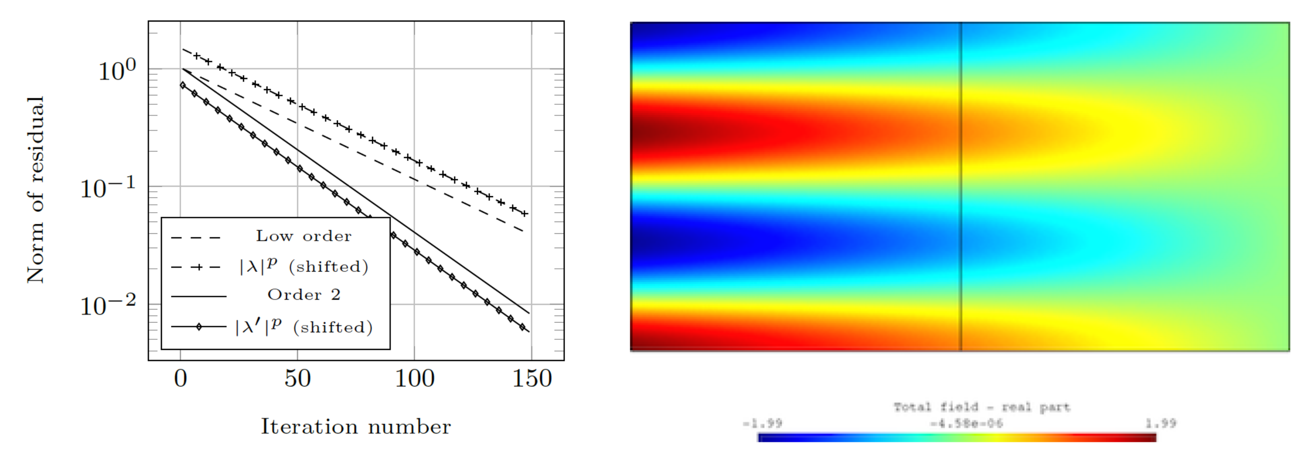
The corner setting ($d_r = 2$) fits in this frame, and one can prove that the set of admissible matrices is not empty for cross points ($d_r \ge 3$).



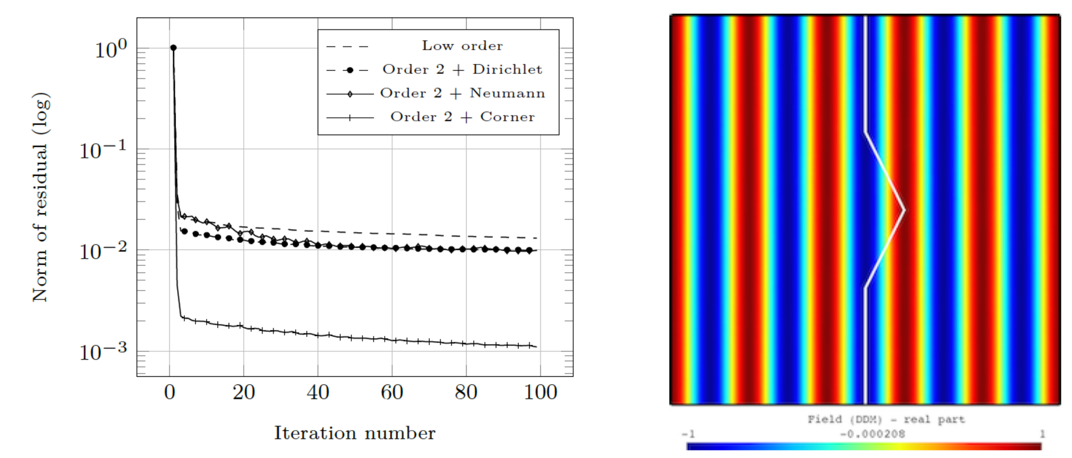
different ABCs vs reference solution



DDM with new ABC vs monodomain sol.



DDM with new TC sanity check



DDM with different TCs (left) new TC (right)